Section 6.1

Laplace Transform

- The (bilateral) Laplace transform of the function x, denoted $\mathcal{L}\{x\}$ or X, is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt.$$

- The inverse Laplace transform of X, denoted $\mathcal{L}^{-1}\{X\}$ or x, is then given by

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds,$$

where $\mathrm{Re}\{s\} = \sigma$ is in the ROC of X. (Note that this is a *contour integration*, since s is complex.)

- We refer to x and X as a Laplace transform pair and denote this relationship as

$$x(t) \overset{\text{LT}}{\longleftrightarrow} X(s).$$

- In practice, we do not usually compute the inverse Laplace transform by directly using the formula from above. Instead, we resort to other means (to be discussed later).

- Two different versions of the Laplace transform are commonly used:
  1. the *bilateral* (or *two-sided*) Laplace transform; and
  2. the *unilateral* (or *one-sided*) Laplace transform.
- The unilateral Laplace transform is most frequently used to solve systems of linear differential equations with nonzero initial conditions.
- As it turns out, the only difference between the definitions of the bilateral and unilateral Laplace transforms is in the *lower limit of integration*.
- In the bilateral case, the lower limit is $-\infty$, whereas in the unilateral case, the lower limit is .0
- For the most part, we will focus our attention primarily on the bilateral Laplace transform.
- We will, however, briefly introduce the unilateral Laplace transform as a tool for solving differential equations.
- Unless otherwise noted, all subsequent references to the Laplace transform should be understood to mean *bilateral* Laplace transform.

- Let X and X_F denote the Laplace and (CT) Fourier transforms of x, respectively.
- The function $X(s)$ evaluated at $s = j\omega$ (where ω is real) yields $X_F(\omega)$. That is,

$$X(s)|_{s=j\omega} = X_F(\omega).$$

- Due to the preceding relationship, the Fourier transform of x is sometimes written as $X(j\omega)$.
- The function $X(s)$ evaluated at an arbitrary complex value $s = \sigma + j\omega$ (where $\sigma = \mathrm{Re}\{s\}$ and $\omega = \mathrm{Im}\{s\}$) can also be expressed in terms of a Fourier transform involving x. In particular, we have

$$X(s)|_{s=\sigma+j\omega} = X'_F(\omega),$$

where X'_F is the (CT) Fourier transform of $x'(t) = e^{-\sigma t}x(t)$.

- So, in general, the Laplace transform of x is the Fourier transform of an exponentially-weighted version of x.
- Due to this weighting, the Laplace transform of a signal may exist when the Fourier transform of the same signal does not.

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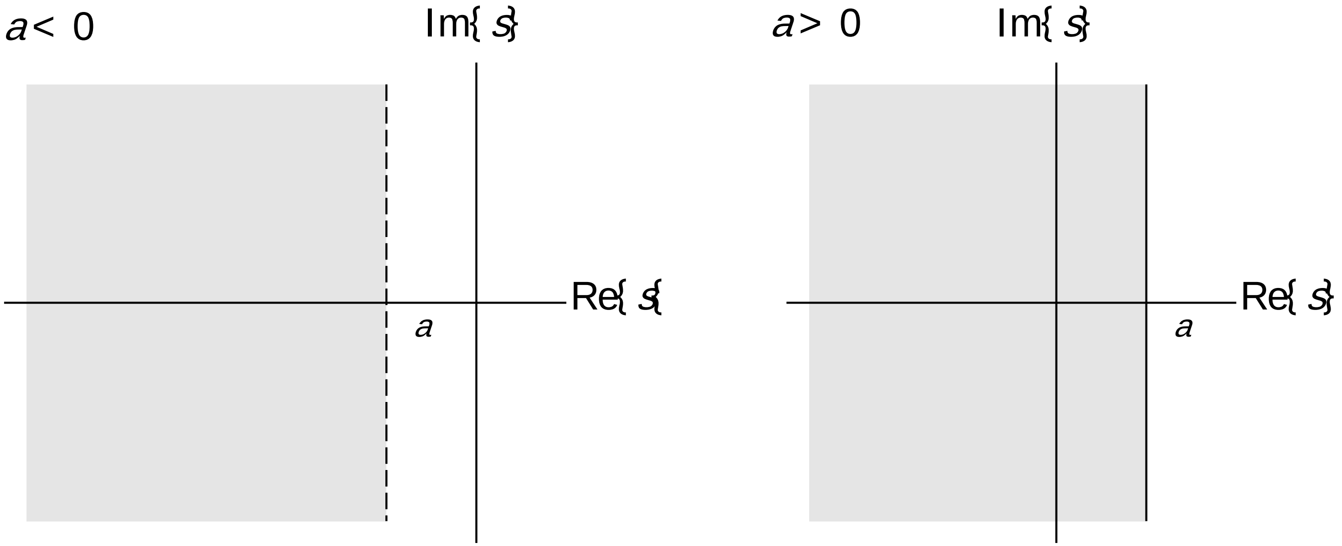
Section 6.2

Region of Convergence (ROC(

- The set R of all complex numbers s satisfying

$$\operatorname{Re}\{s\} < a$$

  for some real constant a is said to be a left-half plane (LHP).
- Some examples of LHPs are shown below.



- The set R of all complex numbers s satisfying

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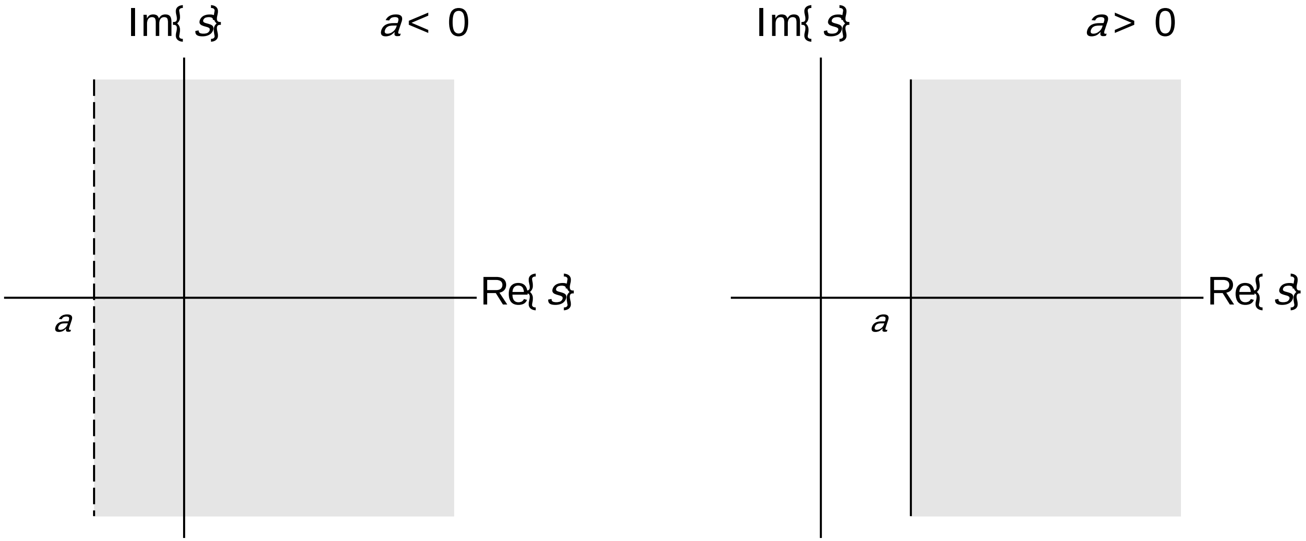
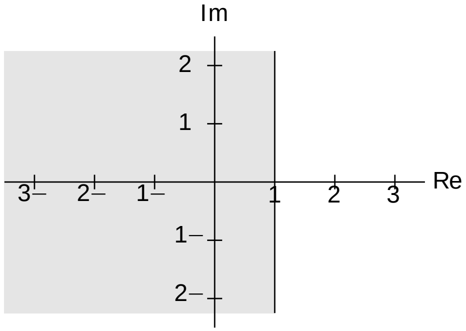
for some real constant a is said to be a right-half plane (RHP).

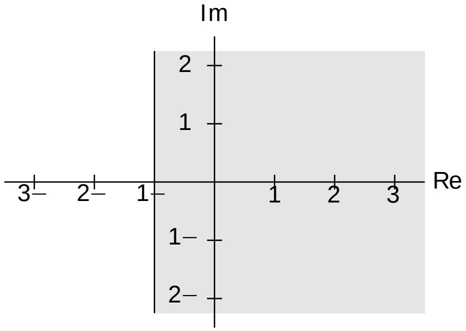
- Some examples of RHPs are shown below.



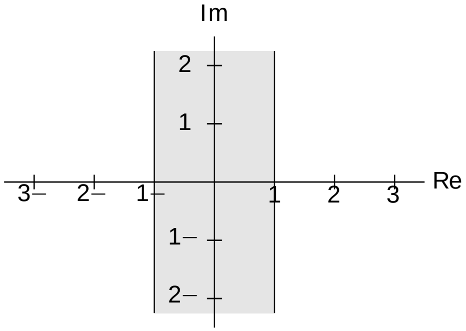
- For two sets A and B, the intersection of A and B, denoted $A \cap B$, is the set of all points that are in both A and B.
- An illustrative example of set intersection is shown below.



R_1



R_2



$R_1 \cap R_2$

- As we saw earlier, for a signal x, the complete specification of its Laplace transform X requires not only an algebraic expression for X, but also the ROC associated with X.
- Two very different signals can have the same algebraic expressions for X.
- Now, we examine some of the constraints on the ROC (of the Laplace transform) for various classes of signals.

1. The ROC of the Laplace transform X consists of *strips parallel to the imaginary axis* in the complex plane.
2. If the Laplace transform X is a *rational* function, the ROC *does not contain any poles*, and the ROC is *bounded by poles or extends to infinity*.
3. If the signal x is *finite duration* and its Laplace transform $X(s)$ converges for some value of s, then $X(s)$ converges for *all values* of s (i.e., the ROC is the entire complex plane).
4. If the signal x is *right sided* and the (vertical) line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}\{x\}$, then all values of s for which $\mathrm{Re}\{s\} > \sigma_0$ must also be in the ROC (i.e., the ROC contains a *RHP* including $\mathrm{Re}\{s\} = \sigma_0$).
5. If the signal x is *left sided* and the (vertical) line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}\{x\}$, then all values of s for which $\mathrm{Re}\{s\} < \sigma_0$ must also be in the ROC (i.e., the ROC contains a *LHP* including $\mathrm{Re}\{s\} = \sigma.(_0$

6. If the signal x is *two sided* and the (vertical) line $\operatorname{Re}\{s\} = \sigma_0$ is in the ROC of the Laplace transform $X = \mathcal{L}\{x\}$, then the ROC will consist of a *strip* in the complex plane that includes the line $\operatorname{Re}\{s\} = \sigma_0$.
7. If the Laplace transform X of the signal x is *rational* (with at least one pole), then:
    1. If x is *right sided*, the ROC of X is to the right of the rightmost pole of X (i.e., the *RHP* to the *right of the rightmost pole*).
    2. If x is *left sided*, the ROC of X is to the left of the leftmost pole of X (i.e., the *LHP* to the *left of the leftmost pole*).

- Some of the preceding properties are *redundant* (e.g., properties 1, 2, 4, and 5 imply property 7).

- Since every function can be classified as one of finite duration, left sided but not right sided, right sided but not left sided, or two sided, we can infer from properties 3, 4, 5, and 6 that the ROC can only be of the form of a LHP, RHP, vertical strip, the entire complex plane, or the empty set. Thus, the ROC must be a *connected region*.

Section 6.3

Properties of the Laplace Transform

Property	Time Domain	Laplace Domain	ROC
Linearity	$a_1x_1(t)+a_2x_2(t)$	$a_1X_1(s)+a_2X_2(s)$	At least $R_1\cap R_2$
Time-Domain Shifting	$x(t-t_0)$	$e^{-st_0}X(s)$	R
Laplace-Domain Shifting	$e^{s_0t}x(t)$	$X(s-s_0)$	$R+\mathrm{Re}\{s_0\}$
Time/Frequency-Domain Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	aR
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Time-Domain Convolution Time-	$x_1*x_2(t)$	$X_1(s)X_2(s)$	At least $R_1\cap R_2$
Domain Differentiation Laplace-	$\frac{d}{dt}x(t)$	$sX(s)$	At least R
Domain Differentiation Time-	$-tx(t)$	$\frac{d}{ds}X(s)$	R
Domain Integration	$\int_{-\infty}^{t}x(\tau)\,d\tau$	$\frac{1}{s}X(s)$	At least $R\cap\{\mathrm{Re}\{s\}>0\}$

Property	
Initial Value Theorem	$x(0^+)=\lim_{s\to\infty}sX(s)$
Final Value Theorem	$\lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)$

Pair	$x(t)$	$X(s)$	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}\{s\} < 0$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} > 0$
5	$-t^n u(-t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}\{s\} < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}\{s\} < -a$
8	$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} > -a$
9	$-t^n e^{-at}u(-t)$	$\frac{n!}{(s+a)^{n+1}}$	$\mathrm{Re}\{s\} < -a$
10	$[\cos\omega_0 t]u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
11	$[\sin\omega_0 t]u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}\{s\} > 0$
12	$[e^{-at}\cos\omega_0 t]u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$
13	$[e^{-at}\sin\omega_0 t]u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}\{s\} > -a$